Vector Identity 19

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

Proof

Note the following vector identity.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

Take the dot product of a constant vector $\mathbf{M}$ with the volume integral in question.

$$\begin{aligned}
\mathbf{M} \cdot \int_V \nabla \times \mathbf{A}\, d^3x &= \int_V \mathbf{M} \cdot (\nabla \times \mathbf{A})\, d^3x \\
&= \int_V \nabla \cdot (\mathbf{A} \times \mathbf{M})\, d^3x \\
&= \int_S (\mathbf{A} \times \mathbf{M}) \cdot \mathbf{n}\, da \\
&= \int_S \mathbf{n} \cdot (\mathbf{A} \times \mathbf{M})\, da \\
&= \int_S \mathbf{M} \cdot (\mathbf{n} \times \mathbf{A})\, da \\
&= \mathbf{M} \cdot \int_S (\mathbf{n} \times \mathbf{A})\, da
\end{aligned}$$

Therefore,

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da.$$